647510 **MSC1MatC104x** **Seat No :________**

M.Sc.(Maths.) Semester - 1 (*CBCS*) Examination
Nov/Dec -2022 [NEW COURSE]
Theory of Ordinary Differential Equations(Core (New))

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Que. 1 (A) Answer any Two out of Three **[10]**

(1) State and prove Abel's formula.

(2) Show that a solution matrix $\phi(t)$ of $y' = A(t)y$ on an interval I is a fundamental matrix of $y' = A(t)y$ in I if and only if $det(\phi(t)) \neq 0$ for every t on I.

(3) If the complex $n \times n$ matrix $A(t)$ is continuous on an interval I, then show that the solution of the system $y' = A(t)y$ on I form a vector space of dimension n over the complex numbers.

(B) Answer any One out of Two **[04]**

(1) If $A(t), g(t)$ are continuous on some interval $a \leq t \leq b$, if $a \leq t_0 \leq b$ and if $|\eta| < \infty$, then prove that the system $y' = A(t)y + g(t)$ has a unique solution $\phi(t)$ satisfying the initial condition $\phi(t_0) = \eta$ and existing on the interval $a \leq t \leq b$.

(2) Show that $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is a fundamental matrix for the system $y' = Ay$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Que. 2 (A) Answer any Two out of Three **[10]**

(1) Find a fundamenta; matrix of the system $y' = Ay$ if A is a diagonal matrix $A = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix}$.

(2) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

(3) Find a fundamental matrix for the system $y' = Ay$ with

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

(B) Answer any One out of Two **[04]**

(1) Define exponential of matrix, Eigen values, Eigen vector of matrices and fundamental matrix.

(2) Prove that the matrix $\phi(t) = exp(At)$ is the fundamental matrix of $y' = Ay$ with $\phi(0) = E$ on $-\infty < t < \infty$.

Que. 3 (A) Answer any Two out of Three **[10]**

(1) Solve $y' - y = 0$ by using method of power series.

(2) Solve Legendre's differential equation $(1 - x^2)y'' - 2xy^{\cdot} + n(n-1)y = 0$, where n is constant.

(3) Solve the ordinary differential equation $x(x-1)y'' + (3x-1)y' + y = 0$.

(B) **Answer any One out of Two** **[04]**

(1) Define radius of convergence. Prove that $J_{-n}(x) = (-1)^n J_{-n}(x)$.

(2) Define regular point and singular point of the ordinary differential equation with example.

Que. 4 **(A)** **Answer any Two out of Three** **[10]**

(1) Let f and $\frac{\partial f}{\partial y}$ are continuous on the closed rectangle R and satify the bounds $|f(t,y)| \leq M, \left|\frac{\partial f}{\partial y}(t,y)\right| \leq K$ $(M > 0, K > 0)$. Prove that the successive approximations ϕ_j, given by $\phi_0(t) = y_0, \phi_{j+1}(t) = y_0 + \int_{t_0}^{t} f\big(s, \phi_j(s)\big)ds, (j = 0,1,2, \dots)$ converge (uniformly) on the interval $I = \{t: |t - t_0| \leq \alpha\}$, to a solution ϕ of the differential equation $y' = f(t,y)$ that satisfies the initial conditions $\phi_0(t) = y_0$.

(2) Suppose f and $\frac{\partial f}{\partial y_j}(j = 1,2,3, \dots, n)$ are continuous on the "box" $B = \{(t,y)/|t - t_0| \leq a, |y - \eta| \leq b\}$. Show that there exists at most one solution of $y' = f(t,y)$ satisfying the initial conditions $\phi_0(t) = \eta$.

(3) Let f is continuous on the rectangle $R = \{(t,y)/|t - t_0| \leq a, |y - y_0| \leq b\}$, and monotone nonincreasing in y for each fixed t on the rectangle R. Prove that the initial value problem $y' = f(t,y), \phi_0(t) = y_0$ has at most one solution on any interval J with t_0 as left end point.

(B) **Answer any One out of Two** **[04]**

(1) Let ϕ is a solution of the initial value problem $y' = f(t,y), \phi_0(t) = y_0$ on an interval I. Show that $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$ on I. Conversely, if $y(t)$ is a solution of $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$ on some interval J containing t_0, then prove that $y(t)$ satifies $y' = f(t,y)$ on J and also the initial condition $\phi_0(t) = y_0$.

(2) Let f and $\frac{\partial f}{\partial y_j}(j = 1,2,3, \dots, n)$ are continuous in a domain D and $|f|$ is bounded in D. Let ϕ be a solution of $y' = f(t,y), \phi_0(t) = \eta$ existing on a finite interval $\gamma < t < \delta$. Show that $\lim_{t \to \delta^-} \phi(t)$ and $\lim_{t \to \gamma^+} \phi(t)$exist.

Que. 5 **(A)** **Answer any Two out of Three** **[10]**

(1) a) Show that $\mathrm{L}\{\mathrm{t}^{\mathrm{n}}\} = \frac{\Gamma_{\mathrm{n+1}}}{\mathrm{s}^{\mathrm{n+1}}}$ for $\mathrm{n} > -1$

b) If $L\{f(t)\} = \bar{f}(s)$and $\{g(t)\} = \bar{g}(s)$, then for any constants a and b $L\{af(t) + b\, g(t)\} = a\, L\{f(t)\} + bL\{g(t)\} = a\bar{f}(s) + b\bar{g}(s)$.

(2) Find the inverse Laplace Transform of $\frac{4s+5}{(s-1)^2(s+2)}$.

(3) Solve differential equations $y'' + 3y' + 2y = e^t$; $y(0) = 1; y'(0) = 0$ using Laplace transform.

(B) **Answer any One out of Two** **[04]**

(1) Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction $|f(t)| \leq Me^{kt}$. Show that for $s > 0, s > k$ and $t > 0$, $L\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}F(s)$ thus $\int_0^t f(x)dx = L^{-1}\left\{\frac{1}{s}F(s)\right\}$.

(2) State and prove convolution theorem.

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